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On the Density Distribution in a Self-Gravitating Liquid Sphere of Constant Compressibility

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In a recent paper the first author has shown that a spherical gravitating liquid planet without spin and having uniform temperature cannot be stable unless the Adams–Williamson condition relating density distribution and compressibility holds throughout the sphere. It is assumed that spherical symmetry obtains, in that the compressibility and density are functions only of t stance from the center of the sphere. Now although constant compressibility is not to be expected as a general feature, it seems a basic question to consider what the distribution of density in such a sphere would be if the compressibility were constant. This paper gives the density distribution as well as the gravity acceleration distribution. The latter is found to exhibit some interesting features.

1. INTRODUCTION

In 1963 the first author [6] showed that for an earth model having a liquid core, we cannot solve the statistical equations for deformation under surface mass loads, unless the Adams–Williamson [1] condition holds in the liquid core. Since then a number of papers have appeared in the literature, some of them supporting this contention, and some disputing it. A brief survey of this literature has recently been given by Longman [9] in a paper where a definitive mathematical proof is given that without the Adams–Williamson condition a self-gravitating liquid planet (without spin and temperature effects) cannot be stable. Physical reasons for the requirement were also suggested.

The present paper starts from the Adams–Williamson condition as applied to a liquid, and assuming constant compressibility calculates the density distribution as well as the distribution of the acceleration due to gravity. Various methods of calculation are compared, and a discussion is given of some interesting features of the numerical results. These are presented in tabular and graphical form.

2. Theory

As noted by the first author [6, 7, 9], the Adams–Williamson condition in a liquid having Lamé elastic parameters $\mu = 0$ and λ can be expressed in the form

$$g + \frac{\lambda}{\rho^2}\frac{d\rho}{dr} = 0. \tag{1}$$

Here $\rho = \rho(r)$ is the density given as a function of distance r from the center of the sphere, while $g = g(r)$ is the "downward" acceleration due to gravity, and is given in terms of ρ by

$$g(r) = \frac{4\pi G}{r^2}\int_0^r \rho(s)\, s^2\, ds, \tag{2}$$

where G is the gravitational constant. Combining (1) and (2) we have then for $\rho(r)$ the nonlinear integro-differential equation

$$\frac{d\rho}{dr} = -\frac{4\pi G}{\lambda}\frac{\rho^2}{r^2}\int_0^r \rho(s)\, s^2\, ds. \tag{3}$$

Till now the Lamé parameter has possibly been a function of position. But now we fix ideas and consider the special but presumably basic case where λ is constant, and this assumption is maintained throughout the remainder of this paper.

It is convenient to cast Eq. (3) into a dimensionless form which also serves as a single-standard equation for the problem. As a first step we make the transformation

$$\rho(r) = \rho_0 y(r), \tag{4}$$

where $\rho_0 = \rho(0)$ is the density at the center of the sphere. $y(r)$ is thus a dimensionless representation of the density distribution. Equation (3) now takes the form

$$\frac{dy}{dr} = -\frac{1}{A^2}\frac{y^2}{r^2}\int_0^r y(s)\, s^2\, ds, \tag{5}$$

where

$$A = \left[\frac{\lambda}{4\pi G\rho_0^2}\right]^{1/2}, \tag{6}$$

and A is easily seen to have the dimension of length. In order to complete the transformation to dimensionless form we now put

$$r = Ax, \tag{7}$$

so that x is a dimensionless representation of distance from the center. Our equation now takes the form

$$\frac{dy}{dx} = -\frac{y^2}{A^3x^2}\int_0^{Ax} y(s/A)\, s^2\, ds, \tag{8}$$

where we now consider $y = y(x)$, instead of $y = y(r)$. Finally we transform the integral by

$$s = Au, \qquad ds = Adu, \tag{9}$$

and obtain our equation in standard form

$$\frac{dy}{dx} = -\frac{y^2}{x^2}\int_0^{x} y(u)\, u^2\, du. \tag{10}$$

Here, of course, the variable of integration u is also a dimensionless representation of the radius. Equation (10) is to be solved subject to the initial conditions

$$y(0) = 1, \tag{11}$$

$$\frac{dy}{dx}(0) = 0. \tag{12}$$

Condition (11) is evident from the definition of y, while (12) follows from the requirement of zero gravity at the center of the sphere. It is worthwhile at this stage to cast Eq. (2) for the gravity distribution also into dimensionless form. We have

$$z(x) = \frac{1}{x^2}\int_0^{x} y(u)\, u^2\, du, \tag{13}$$

where we have written

$$g(r) = Bz(x) \tag{14}$$

with

$$B = 4\pi GA\rho_0 = [4\pi\lambda G]^{1/2}, \tag{15}$$

which has the dimension of acceleration. Thus $z(x)$ is a dimensionless form of the "downward" acceleration due to gravity in the sphere.

Our task therefore, is now, to solve Eq. (10) subject to the initial conditions (11) and (12), and then to compute $z(x)$ as given by (13). This has been carried out in several ways, and these are described in the next section. However, some general features of the problem are first described here.

The behavior of the solution as $x \to \infty$ is of some interest. In the first place it is readily verified by substitution that

$$y = \sqrt{2}/x \tag{16}$$

is an exact solution of (10), but it does *not* satisfy the boundary conditions (11), (12). Furthermore, as noted below in Section 4, (16) seems to approximate the solution to our problem for large x. It is readily verified by L'Hospital's rule that if $\lim_{x\to\infty} z(x)$ exists, then the value of this limit is $1/\sqrt{2}$. For suppose

$$l = \lim_{x\to\infty} \left[\int_0^x y(u)\, u^2\, du\right] \Big/ x^2. \tag{17}$$

Then by L'Hospital,

$$l = \lim_{x\to\infty} \frac{yx^2}{2x} = \lim_{x\to\infty} \frac{x}{2/y} = \lim_{x\to\infty} \left(-\frac{2}{y^2}\frac{dy}{dx}\right)^{-1} = \frac{1}{2l},$$

by (10). Thus $l^2 = \frac{1}{2}$ and so

$$l = 1/\sqrt{2}. \tag{18}$$

We now see from (10) that for large x we have approximately

$$\frac{d}{dx}\left(\frac{1}{y}\right) = \frac{1}{\sqrt{2}}$$

and so

$$1/y = (x/\sqrt{2}) + c$$

approximately, for large x, where a is a constant. It follows that indeed

$$\lim_{x\to\infty} xy = \sqrt{2}, \tag{19}$$

as suggested by the numerical results.

It is of interest to note that these results on the behavior of $y(x)$ and $z(x)$ have been obtained independently of the initial conditions (11), (12).

It is instructive also to express (10) as an ordinary differential equation. We readily derive

$$\frac{d^2w}{dx^2} + \frac{2}{x}\frac{dw}{dx} - \frac{1}{w} = 0, \tag{20}$$

where

$$w = 1/y. \tag{21}$$

Equation (20) is an Emden-type equation [3, p. 381], the general form of which is

$$\frac{d^2w}{dx^2} + \frac{2}{x}\frac{dw}{dx} + f(w) = 0, \tag{22}$$

and in our case

$$f(w) = -1/w. \tag{23}$$

The case

$$f(w) = w^n, \qquad n = 0, 1, 2, 3,... \tag{24}$$

gives rise to Emden's equation [3, p. 371] which was first studied by the German astrophysicist Emden [4] in his work on the thermal behavior of a spherical cloud of gas under the mutual attraction of its molecules and subject to the classical laws of thermodynamics. The solutions of most interest are the Emden functions, which have been studied extensively and tabulated by the British Association for the Advancement of Science [2].

The case (23), however, has not to the authors' knowledge been previously considered. Of interest, however, is an alternative form of the Emden-type equation in which the first-derivative term is absent. This is achieved [3, p. 382] by making the substitution

$$w = \xi/x, \tag{25}$$

and in our case leads to the equation

$$\xi \frac{d^2\xi}{dx^2} = x^2, \tag{26}$$

which is to be solved together with the boundary conditions

$$\xi(0) = 0, \tag{27}$$

$$\frac{d\xi}{dx}(0) = 1. \tag{28}$$

In terms of our original independent variable we have

$$\xi = x/y. \tag{29}$$

3. Methods of Solution

The first method considered was that of successive approximations which satisfy the initial conditions. The first approximation was taken as

$$y_1 = 1 \tag{30}$$

and substituted in the right-hand side of (10). Two integrations and the use of (11) led us to the second approximation,

$$y_2 = 1 - (x^2/6), \tag{31}$$

and proceeding in this way we found

$$y_3 = 1 - (x^2/6) + (13/360)\,x^4 - (11/3240)\,x^6 + (1/8640)\,x^8, \tag{32}$$

and a few further approximations. However, the nth approximation is a polynomial of degree $3^{n-1} - 1$, so that we rapidly run into computational difficulties in the computer (due to the enormous powers of x that have to be taken into account), even though the process seems to converge well for given x. It is not difficult to see that at each stage the coefficient of one more term becomes fixed. For example the fourth approximation (a polynomial of degree 26) commences with the terms

$$y_4 = 1 - (x^2/6) + (13/360)\,x^4 - (25/3024)\,x^6 + \cdots, \tag{33}$$

and the last coefficient given above is now the permanent coefficient of x^6 in all the later approximations. This suggests that we can readily obtain a Maclaurin series expansion for $y(x)$, and this is easily done by writing (10) in the form

$$\frac{d}{dx}\left(\frac{1}{y}\right) = \frac{1}{x^2}\int_0^x y(u)\,u^2\,du, \tag{34}$$

substituting a power series for y (from which the reciprocal series is readily obtainable, for example recursively by the method given by Longman [8], and comparing coefficients. The resulting series, however, which may be regarded as the limiting form of y_n as $n \to \infty$, does not seem to converge for x larger than 2. However, very rapid convergence can be induced for values of x to about 3.8 by applying the T-transformation due to the second author [5]. A second T-transformation enabled the range to be extended to about $x = 9$. Beyond this, difficulties were experienced due to loss of accuracy.

The best results for large x were obtained in the computer by the Runge–Kutta numerical integration process. For this purpose we define

$$y_1 = y, \tag{35}$$

$$y_2 = \int_0^x y(u)\,u^2\,du, \tag{36}$$

and our Runge–Kutta equations are

$$\dot{y}_1 = -y_1^2 y_2/x^2, \qquad \dot{y}_2 = y_1 x^2, \tag{37}$$

and these are integrated outward starting from

$$y_1(0) = 1, \qquad y_2(0) = 0. \tag{38}$$

In order to avoid division by zero, at $x = 0$ the equations were (appropriately) replaced by

$$\begin{aligned} \dot{y}_1 &= 0, \\ \dot{y}_2 &= 0, \end{aligned} \qquad \text{at} \quad x = 0. \tag{39}$$

The integration was carried out in the computer, using the IBM subroutine RKGS, in steps of 0.1 till $x = 300$. As an alternative, and a check, Eq. (26) was put in Runge–Kutta form by defining

$$\xi_1 = \frac{d\xi}{dx}, \tag{40}$$

$$\xi_2 = \xi, \tag{41}$$

and then

$$\dot{\xi}_1 = x^2/\xi_2, \qquad \dot{\xi}_2 = \xi_1. \tag{42}$$

Here we started from

$$\xi_1(0) = 1, \qquad \xi_2(0) = 0, \tag{43}$$

and at $x = 0$ the equations were (correctly) replaced by

$$\dot{\xi}_1 = 0, \qquad \dot{\xi}_2 = 1. \tag{44}$$

The computer also printed out

$$y = x/\xi_2 \tag{45}$$

as the solution of (10) with (11), (12).

The accuracy of the Runge–Kutta method for large x may be connected with the fact that for example (16), which satisfies *very different* initial conditions at $x = 0$, nevertheless gives a fairly good approximation to the solution for x larger than, say, 5. For smaller values of x it was difficult to achieve six-figure accuracy, but then other methods were used, as noted above.

4. Numerical Results and Discussion

Table I gives values of y, z for $x = 0(0.1)\,5.0$, while Table II gives the same quantities for $x = 5.0(0.5)\,30$. Results for larger x are presented in Table III for $x = 30(5)\,300$. All results are given to six decimal places, and are believed to be correct to this accuracy.

TABLE I

Values of Dimensionless Density y and Dimensionless Gravity z, in Terms of Dimensionless Radius x, for $x = 0(0.1)5.0$

x	y	z	x	y	z
0	1.000000	0.000000	2.6	0.528930	0.566625
0.1	0.998337	0.033300	2.7	0.513419	0.575626
0.2	0.993391	0.066402	2.8	0.498575	0.584044
0.3	0.985287	0.099112	2.9	0.484375	0.591919
0.4	0.974225	0.131251	3.0	0.470792	0.599287
0.5	0.960468	0.162654	3.1	0.457800	0.606183
0.6	0.944324	0.193177	3.2	0.445374	0.612640
0.7	0.926131	0.222698	3.3	0.433487	0.618688
0.8	0.906239	0.251119	3.4	0.422114	0.624354
0.9	0.884999	0.278366	3.5	0.411229	0.629666
1.0	0.862743	0.304387	3.6	0.400809	0.634646
1.1	0.839785	0.329153	3.7	0.390831	0.639318
1.2	0.816405	0.352651	3.8	0.381271	0.643703
1.3	0.792852	0.374887	3.9	0.372109	0.647820
1.4	0.769339	0.395880	4.0	0.363324	0.651688
1.5	0.746043	0.415658	4.1	0.354898	0.655322
1.6	0.723113	0.434260	4.2	0.346811	0.658739
1.7	0.700663	0.451731	4.3	0.339046	0.661952
1.8	0.678785	0.468120	4.4	0.331587	0.664975
1.9	0.657545	0.483479	4.5	0.324418	0.667821
2.0	0.636990	0.497862	4.6	0.317525	0.670501
2.1	0.617150	0.511323	4.7	0.310893	0.673025
2.2	0.598043	0.523915	4.8	0.304510	0.675404
2.3	0.579674	0.535692	4.9	0.298364	0.677646
2.4	0.562040	0.546703	5.0	0.292442	0.679761
2.5	0.545131	0.556998			

TABLE II

Values of Dimensionless Density y and Dimensionless Gravity z, in Terms of Dimensionless Radius x, for $x = 5.0(0.5)30.0$

x	y	z	x	y	z
5.0	0.292442	0.679761	18.0	0.078982	0.714239
5.5	0.265837	0.688680	18.5	0.076815	0.714050
6.0	0.243439	0.695395	19.0	0.074765	0.713860
6.5	0.224375	0.700482	19.5	0.072822	0.713672
7.0	0.207983	0.704353	20.0	0.070978	0.713485
7.5	0.193760	0.707306	20.5	0.069225	0.713301
8.0	0.181315	0.709560	21.0	0.067558	0.713120
8.5	0.170343	0.711277	21.5	0.065969	0.712943
9.0	0.160604	0.712579	22.0	0.064453	0.712769
9.5	0.151906	0.713557	22.5	0.063006	0.712599
10.0	0.144092	0.714283	23.0	0.061623	0.712433
10.5	0.137037	0.714809	23.5	0.060299	0.712271
11.0	0.130637	0.715178	24.0	0.059032	0.712114
11.5	0.124806	0.715424	24.5	0.057817	0.711960
12.0	0.119472	0.715570	25.0	0.056651	0.711811
12.5	0.114574	0.715640	25.5	0.055314	0.711666
13.0	0.110062	0.715647	26.0	0.054455	0.711525
13.5	0.105892	0.715606	26.5	0.053421	0.711389
14.0	0.102026	0.715526	27.0	0.052425	0.711256
14.5	0.098433	0.715416	27.5	0.051465	0.711128
15.0	0.095086	0.715283	28.0	0.050540	0.711003
15.5	0.091959	0.715132	28.5	0.049648	0.710882
16.0	0.089032	0.714968	29.0	0.048788	0.710765
16.5	0.086286	0.714794	29.5	0.047956	0.710651
17.0	0.083705	0.714613	30.0	0.047153	0.710541
17.5	0.081274	0.714428			

TABLE III

Values of Dimensionless Density y and Dimensionless Gravity z, in Terms of Dimensionless Radius x, for $x = 30(5)300$

x	y	z	x	y	z
30	0.047153	0.710541	170	0.008317	0.706882
35	0.040391	0.709612	175	0.008080	0.706886
40	0.035330	0.708932	180	0.007855	0.706891
45	0.031400	0.708430	185	0.007643	0.706896
50	0.028258	0.708056	190	0.007442	0.706901
55	0.025688	0.707774	195	0.007251	0.706906
60	0.023548	0.707559	200	0.007070	0.706911
65	0.021737	0.707395	205	0.006898	0.706916
70	0.020185	0.707268	210	0.006734	0.706921
75	0.018841	0.707170	215	0.006577	0.706926
80	0.017664	0.707094	220	0.006428	0.706931
85	0.016626	0.707035	225	0.006285	0.706935
90	0.015703	0.706990	230	0.006148	0.706940
95	0.014877	0.706954	235	0.006017	0.706945
100	0.014134	0.706927	240	0.005892	0.706949
105	0.013461	0.706907	245	0.005772	0.706954
110	0.012850	0.706891	250	0.005657	0.706958
115	0.012292	0.706880	255	0.005546	0.706962
120	0.011780	0.706872	260	0.005439	0.706966
125	0.011309	0.706867	265	0.005336	0.706970
130	0.010874	0.706864	270	0.005238	0.706974
135	0.010472	0.706863	275	0.005142	0.706978
140	0.010098	0.706863	280	0.005051	0.706982
145	0.009750	0.706864	285	0.004962	0.706985
150	0.009425	0.706867	290	0.004876	0.706989
155	0.009122	0.706870	295	0.004794	0.706992
160	0.008837	0.706873	300	0.004714	0.706996
165	0.008569	0.706877			

Some results are also given in graphical form. Figure 1 shows a plot of $y(x)$ for $0 \leqslant x \leqslant 9$. Also in this figure is shown the behavior of the function $\sqrt{2}/x$ (indicated by crosses). As remarked in Section 2, this function approximates y for large x. Figure 2 shows a plot of $z(x)$ for $0 \leqslant x \leqslant 36$, and the curve appears to be approaching its asymptote $z = 1/\sqrt{2}$. However, when the results for $10 \leqslant x \leqslant 290$ are plotted with an enlarged z-scale (Fig. 3) it is evident that $z(x)$ goes through a (very flat) minimum before approaching the asymptote. There is also a maximum value of $z(x)$ which appears very flat in Fig. 2, but which is very evident in Fig. 3 where the enlarged z-scale is employed. The maximum and minimum are located approximately at $x = 12.825$ and 136.6 respectively, and we have the following values of y, z.

x	y	z
12.825	0.111600	0.715650
136.6	0.010349	0.706863.

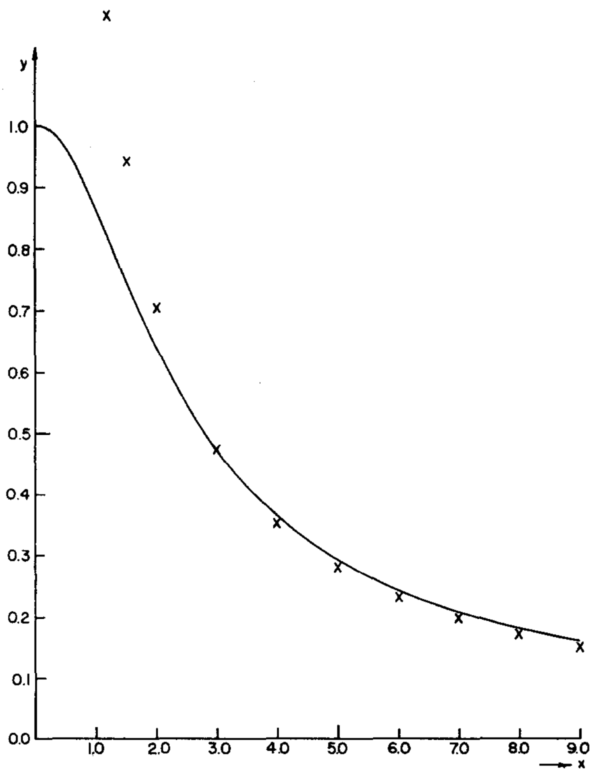

FIG. 1. Dimensionless density y plotted against dimensionless radial distance x from the center of the sphere. The function $y = \sqrt{2}/x$ (indicated by crosses) is shown for comparison.

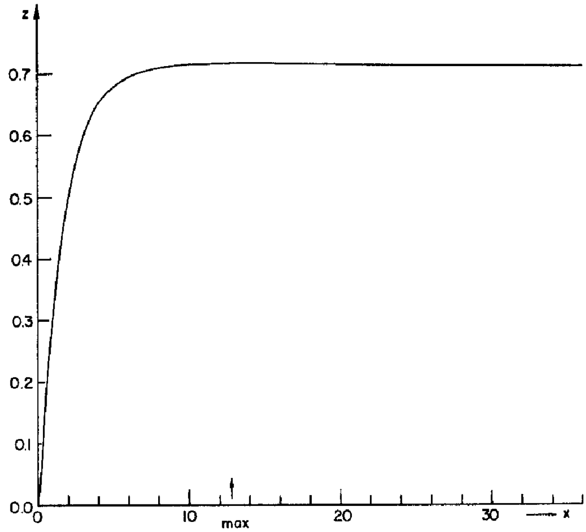

FIG. 2. Dimensionless gravity z plotted against dimensionless distance x from the center of the sphere.

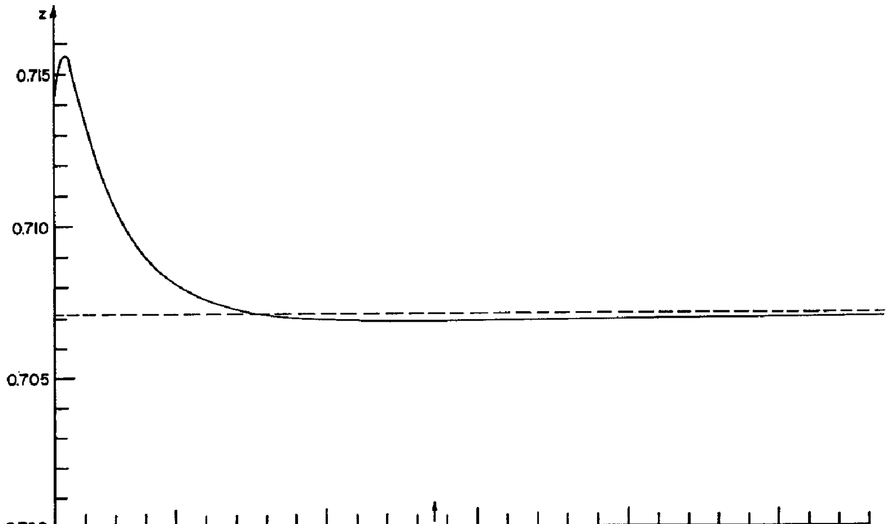

FIG. 3. Dimensionless gravity z plotted against dimensionless distance x from the center of the sphere.

It would seem that the existence of a maximum and a minimum of $g(r)$ is of some physical interest in this basic problem where the Lamé parameter λ is a constant, but the authors have not considered this further at the present time. Also the fact that $g(r)$ approaches asymptotically a finite value as r increases is presumably not without physical interest.

5. Conclusion

The density and gravity distribution in a self-gravitating liquid sphere of constant compressibility, in which the Adams–Williamson condition holds, have been calculated. The results in tabular and graphical form exhibit some interesting features.

Acknowledgments

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